对称矩阵　① 对称线性变换

$A \in M_n(\mathbb{R})$　　$\langle Ax, y\rangle = \langle x, Ay\rangle$

$A = A^T$　　$\Rightarrow$ 正交对角化.

$$Q^{-1}AQ = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

$QQ^T = I_n$.

$Q^{-1} = Q^T$, $Q^TAQ = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$

② B 对称双线性型.

Gram 矩阵. P 可逆

$$P^TAP = \begin{bmatrix} 1 & & & & & & & & \\ & \ddots & & & & & & & \\ & & 1 & & & & & & \\ & & & -1 & & & & & \\ & & & & \ddots & & & & \\ & & & & & -1 & & & \\ & & & & & & 0 & & \\ & & & & & & & \ddots & \\ & & & & & & & & 0 \end{bmatrix}$$

signature (p, q, r)

p 个 1. q 个 -1. r 个 0.

正定性: $B(v, v) > 0$, $\forall v \neq 0$

$x^TAx > 0$. $\forall x \in \mathbb{R}^n$. $x \neq 0$

性质: 以下等价. ① A 正定

② A 的特征值均 > 0.

③ $A = P^TP$, P 可逆.

性质: A 半正定 $\Leftrightarrow$ $A = Q^T \cdot Q$. $\underset{\text{某个}}{Q}$ $m \times n$ 矩阵.

"$\Rightarrow$" $\quad P^T A P = \underbrace{\begin{bmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{bmatrix}}$ $= B$

$$B = B \cdot B, \quad B^T = B$$

$$A = (P^{-1})^T \cdot B^T \cdot B \cdot P^{-1}$$

$$= (B P^{-1})^T \cdot (B P^{-1})$$

"$\Leftarrow$" $\quad x^T A x = x^T \cdot Q^T \cdot Q x$

$$= \underline{(Qx)}^T \cdot \underline{(Qx)} \geq 0.$$

$\underline{\forall x \in \mathbb{R}^n.}$ $\quad A$ 半正定.

练习: A 正定 $\Leftrightarrow$ A 的顺序主子式 > 0 (归纳)

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix} \qquad \left| \begin{bmatrix} 1 \end{bmatrix} \right| = 1 > 0$$

$$\left| \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \right| = 0$$

$$|A|$$

奇异值分解. (singular value decomposition)

SVD

A $m\times n$ 实矩阵. 存在. P, Q, D. 使得

定理(定义): $A = QDP^T$. $Q\in O(m)$

(SVD) $P\in O(n)$

$$D = \left[\begin{array}{cccc} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_n \\ \hline & & 0 & \end{array}\right] \quad m\geq n$$

$$D = \left[\begin{array}{cccc|c} \sigma_1 & & & & \\ & \sigma_2 & & & \\ & & \ddots & & 0 \\ & & & \sigma_m & \end{array}\right] \quad m\leq n$$

$\sigma_i\geq 0$. 通常取 $\sigma_1\geq\sigma_2\cdots\geq\sigma_m\geq 0$

σ_i 称为 A 的奇异值.

回忆: A 相抵 $\Leftrightarrow$ 取不同基下. A 视为线性映射的矩阵表示.

$$A: \mathbb{R}^n \to \mathbb{R}^m.$$
$$x \mapsto A\cdot x$$

$\mathbb{R}^n$ 一组基 B: $\underline{v_1 \cdots v_n}$ $\quad \underline{(v_1 \cdots v_n) = P}$

$\mathbb{R}^m$ 一组基 C: $w_1 \cdots w_m$ $\quad (w_1 \cdots w_m) = Q$

$$\underline{A\cdot(v_1 \cdots v_n)} = (A\cdot v_1 \cdots A v_n)$$
$$= \underline{([Av_1]_C, \cdots [Av_n]_C)}$$
$$\boxed{Q^{-1}A\cdot P}.$$

(SVD)

$$A = Q D\cdot P^T \qquad \underline{P^T = P^{-1}}$$

$$Q^{-1}A\cdot P = D$$

想要取两组基. $\underline{P = (v_1 \cdots v_n)}$ $\mathbb{R}^n$ 中标准正交基

$Q = (w_1 \cdots w_m)$ $\mathbb{R}^m$ 中标准正交基.

$(m \geq n)$ $A \cdot (v_1 \cdots v_n) = (w_1 \cdots w_m) \cdot \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \\ & 0 & \end{bmatrix}$

$$\boxed{A v_i = \sigma_i \cdot w_i.}$$

证明：考虑 $A^T A$ 对称，半正定． $n \times n$．

$A^T A$ 可以正交对角化．

存在 $P \in O(n)$． $P = (v_1 \cdots v_n)$

$$P^{-1} A^T A P = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}, \quad \lambda_i \geq 0$$

$\lambda_1 \geq \lambda_2 \cdots \geq \lambda_n$

$$\boxed{A^T A v_i = \lambda_i v_i}$$

Claim: $A v_1, \cdots A v_n$ 相互正交．

$i \neq j$． $\langle A v_i, A v_j \rangle_{\mathbb{R}^m} = (A v_i)^T \cdot (A v_j)$

$$= v_i^T (A^T A v_j)$$

$$= v_i^T \cdot (\lambda_j v_j)$$

$$= \lambda_j \langle v_i, v_j \rangle_{\mathbb{R}^n} = 0.$$

假设 $\lambda_1 \cdots \lambda_s > 0$. $\boxed{\lambda_{s+1} = \cdots = \lambda_n = 0}$

$$w_1 = \frac{Av_1}{|Av_1|}, \quad w_2 = \frac{Av_2}{|Av_2|} \quad \cdots \quad w_s = \frac{Av_s}{|Av_s|}$$

$w_1 \cdots w_s$ 标准正交，扩充为 $w_1 \cdots w_m$ $\mathbb{R}^m$ 的标准正交基.

$$\boxed{Av_i = |Av_i| \cdot w_i.} \quad 1 \le i \le s$$

$i \ge s+1$

$$\boxed{A^T A v_i = 0} \Rightarrow \langle v_i, A^T A v_i \rangle_{\mathbb{R}^n} = v_i^T \cdot A^T A v_i$$

$$= (Av_i)^T \cdot (Av_i)$$

$$= \langle Av_i, Av_i \rangle_{\mathbb{R}^m}$$

$$= 0$$

$$\Rightarrow Av_i = 0$$

$$Av_i = 0 = 0 \cdot w_i.$$

$$P = (v_1 \cdots v_n)$$

$$Q = (w_1 \cdots w_m)$$

$$A \cdot P = Q \cdot \begin{bmatrix} |Av_1| & & & & \\ & \ddots & & & \\ & & |Av_s| & & \\ & & & 0 & \\ & & & & \ddots \\ & & & & & 0 \end{bmatrix}$$

$$\langle Av_i, Av_i\rangle_{\mathbb{R}^m} = \langle v_i, \underline{A^TAv_i}\rangle_{\mathbb{R}^n}$$

$$= \langle v_i, \lambda_i v_i\rangle_{\mathbb{R}^n}$$

$$= \lambda_i$$

$$\Rightarrow |Av_i| = \sqrt{\lambda_i} \qquad 1 \le i \le s.$$

$$\boxed{AP = Q \cdot \begin{pmatrix} \sqrt{\lambda_1} \cdots \sqrt{\lambda_n} \\ \hline 0 \end{pmatrix}}$$

$$A = Q \cdot DP^T, \qquad A^TA = PD^TQ^T \cdot QDP^T$$

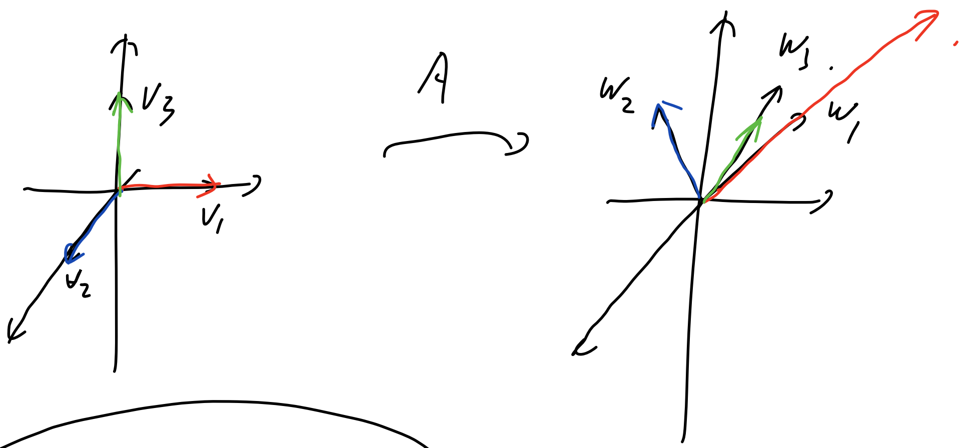
$$= P \cdot \underline{D^TD} \cdot P^T$$

$$D = \begin{bmatrix} \sigma_1 \ddots \sigma_n \\ 0 \end{bmatrix} \qquad D^TD = \begin{bmatrix} \sigma_1^2 & & \\ & \ddots & \\ & & \sigma_n^2 \end{bmatrix}$$

σ_i^2 是 A^TA 的特征值.

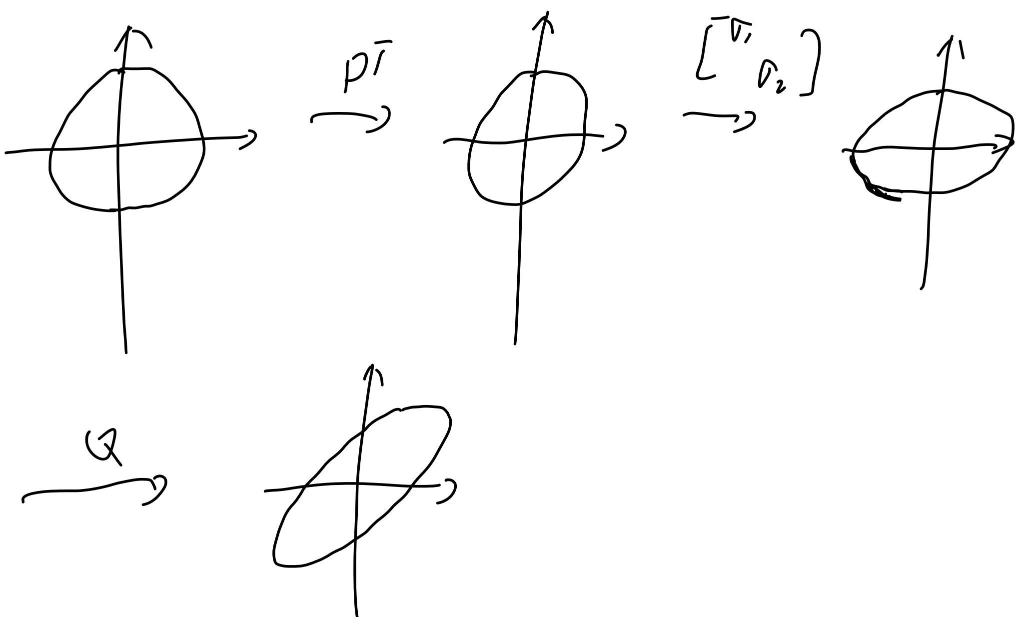
$m \ge n$, $m \le n$ $\begin{cases} A = QDP^T & SVD \\ \Downarrow \\ A^T = PD^TQ^T & SVD \end{cases}$

几何上：



$$A = QDP^T$$

2×2



应用场景

图像压缩. 像素 $m\times n$（灰度，黑白）

$A = (a_{ij})$. a_{ij} (i,j)点的灰度.

低秩逼近.

$m \geq n$

回顾:
$$\underline{A \cdot P} = \underline{Q} \cdot \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \\ & & \end{bmatrix}$$

$$= \begin{bmatrix} w_1 \cdots w_s & \boxed{} \end{bmatrix} \cdot \begin{bmatrix} \lambda_1 & & & & \\ & \ddots & & & \\ & & \lambda_s & & \\ & & & 0 & \\ & & & & \ddots \\ & & & & & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \underline{w_1 \cdots w_s} & 0 \cdots 0 \end{bmatrix} \cdot \begin{bmatrix} \lambda_1 & & & & \\ & \ddots & & & \\ & & \lambda_s & & \\ & & & 0 & \\ & & & & \ddots \\ & & & & & 0 \end{bmatrix}$$

定义: (rank r 逼近) A_r

$$A = Q \cdot \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \\ \hline & 0 & \end{bmatrix} P^T$$

$$A_r = Q \cdot \begin{bmatrix} \sigma_1 & & & & \\ & \ddots & & & \\ & & \sigma_r & & \\ & & & 0 & \\ & & & & \ddots \\ & & & & & 0 \\ & & & & & \end{bmatrix} P^T$$

$$= \underbrace{(Q_1 \quad 0)}_{Q\text{前}r\text{列}} \cdot \left(\begin{array}{c|c} \sigma_1 \cdots \sigma_r & \\ \hline & 0 \end{array}\right) \cdot P^T$$

$$= Q_1 \cdot \begin{pmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_r \end{pmatrix} \cdot P_1^T$$

P_1^T 是 P^T 的前 r 行.

$\mathrm{rk}\, A_r \leq r$

A. A_r "比较接近"

$V = M_{m \times n}(\mathbb{R})$ 上有内积 $\langle \ , \ \rangle$

定义 $\langle A, B \rangle = \mathrm{Tr}(A^T B)$ (Frobenius)

$A = \begin{pmatrix} a & b & e \\ c & d & f \end{pmatrix}$, $B = \begin{pmatrix} a' & b' & e' \\ c' & d' & f' \end{pmatrix}$

$$\mathrm{Tr}(A^T \cdot B) = \mathrm{Tr}\left(\begin{pmatrix} a & c \\ b & d \\ e & f \end{pmatrix} \cdot \begin{pmatrix} a' & b' & e' \\ c' & d' & f' \end{pmatrix}\right)$$

$$= \begin{pmatrix} aa'+cc' & & \\ & bb'+dd' & \\ & & ee'+ff' \end{pmatrix}$$

$$= aa'+bb'+cc'+dd'+ee'+ff'$$

（$\mathbb{R}^{mn}$ 中标准内积一样）

"长度" $\|A\|_F = \sqrt{\mathrm{Tr}(A^T A)}$

A, B 距离 $\mathrm{dist}(A, B) = \|A-B\|_F$

定理：(Eckart-Young, Schmidt)

A_r 是 $\mathrm{rk} \le r$ 的矩阵 ($m \times n$) 中与 A 的 Frobenius

距离最近的矩阵.

$$\|A-A_r\|_F = \min_{B\ \mathrm{rk}\le r} \|A-B\|_F$$

引理: $V = M_{n\times n}(\mathbb{R})$. $\langle , \rangle$. Frobenius 内积.

$Q\in O(n)$ 则 $V\to V$ 是正交变换
$A\mapsto QA$

$P\in O(n)$, 则 $V\to V$ 也是正交变换
$A\mapsto AP$

证明: $\langle QA, QB\rangle = Tr((QA)^T(QB))$

$= Tr(A^T\cdot(Q^TQ)B)$

$= Tr(A^TB)$

$= \langle A, B\rangle$

$\langle AP, BP\rangle = Tr((AP)^T\cdot BP)$

$= Tr(\underline{P^T}\,\underline{A^TBP})$

$= Tr(A^TB\cdot PP^T)$

$= Tr(A^T B)$

$= \langle A, B \rangle$.

推论: $A = QDP^T$. $D = \begin{pmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \\ & 0 & \end{pmatrix}$

$$\|A\|_F = \sqrt{\sigma_1^2 + \cdots + \sigma_n^2}$$

推论: $\|A - A_k\|_F^2 = Q \cdot \begin{pmatrix} 0 & & & & \\ & \ddots & & & \\ & & \sigma_{k+1} & & \\ & & & \ddots & \\ & & & & \sigma_n \\ & & 0 & & \end{pmatrix} P^T$

$$= \sum_{i \geq k+1} (\sigma_i)^2 .$$

lost energy

第一个奇异值 σ_1.

性质: $$\sigma_1 = \max_{\substack{v \in \mathbb{R}^n \\ v \neq 0}} \frac{|A \cdot v|}{|v|}$$

回归 A^TA 的 λ_1.

$$\lambda_1 = \max_{\substack{v \in \mathbb{R}^n \\ |v|=1}} \langle v, A^TAv \rangle$$

$$\langle Av, Av \rangle_{\mathbb{R}^m}.$$

$$/\!/$$

$$= \max_{v \neq 0} \frac{\langle v, A^TAv \rangle_{\mathbb{R}^n}}{\langle v, v \rangle}$$

$\sigma_1 = \sqrt{\lambda_1}$ 二) 性质.

$\boxed{\sigma_2, \cdots \sigma_{\min(m,n)}}$ 也有 min, max 描述